

# 1 SymPy: Symbolic Computing in Python

## 2 Supplementary material

3 As in the paper, all examples in the supplement assume that the following has been run:

```
4 >>> from sympy import *  
5 >>> x, y, z = symbols('x y z')
```

## 6 1 LIMITS: THE GRUNTZ ALGORITHM

7 SymPy calculates limits using the Gruntz algorithm, as described in [7]. The basic idea is as  
8 follows: any limit can be converted to a limit  $\lim_{x \rightarrow \infty} f(x)$  by substitutions like  $x \rightarrow \frac{1}{x}$ . Then  
9 the subexpression  $\omega$  (that converges to zero as  $x \rightarrow \infty$  faster than all other subexpressions) is  
10 identified in  $f(x)$ , and  $f(x)$  is expanded into a series with respect to  $\omega$ . Any positive powers  
11 of  $\omega$  converge to zero (while negative powers indicate an infinite limit) and any constant term  
12 independent of  $\omega$  determines the limit. When a constant term still depends on  $x$  the Gruntz  
13 algorithm is applied again until a final numerical value is obtained as the limit.

To determine the most rapidly varying subexpression, the comparability classes must first be defined, by calculating  $L$ :

$$L \equiv \lim_{x \rightarrow \infty} \frac{\log|f(x)|}{\log|g(x)|} \quad (1)$$

The relations  $<$ ,  $>$ , and  $\sim$  are defined as follows:  $f > g$  when  $L = \pm\infty$  (it is said that  $f$  is more rapidly varying than  $g$ , i.e.,  $f$  goes to  $\infty$  or 0 faster than  $g$ ),  $f < g$  when  $L = 0$  ( $f$  is less rapidly varying than  $g$ ) and  $f \sim g$  when  $L \neq 0, \pm\infty$  (both  $f$  and  $g$  are bounded from above and below by suitable integral powers of the other). Note that if  $f > g$ , then  $f > g^n$  for any  $n$ . Here are some examples of comparability classes:

$$\begin{aligned} 2 < x < e^x < e^{x^2} < e^{e^x} \\ 2 \sim 3 \sim -5 \\ x \sim x^2 \sim x^3 \sim \frac{1}{x} \sim x^m \sim -x \\ e^x \sim e^{-x} \sim e^{2x} \sim e^{x+e^{-x}} \\ f(x) \sim \frac{1}{f(x)} \end{aligned}$$

The Gruntz algorithm is now illustrated with the following example:

$$f(x) = e^{x+2e^{-x}} - e^x + \frac{1}{x}. \quad (2)$$

14 First, the set of most rapidly varying subexpressions is determined — the so-called *mrv set*.  
15 For (2), the mrv set  $\{e^x, e^{-x}, e^{x+2e^{-x}}\}$  is obtained. These are all subexpressions of (2) and they  
16 all belong to the same comparability class. This calculation can be done using SymPy as follows:

```
17 >>> from sympy.series.gruntz import mrv  
18 >>> mrv(exp(x+2*exp(-x))-exp(x) + 1/x, x)[0].keys()  
19 dict_keys([exp(x + 2*exp(-x)), exp(x), exp(-x)])
```

20 Next, an arbitrary item  $\omega$  is taken from mrv set that converges to zero for  $x \rightarrow \infty$  and doesn't  
21 have subexpressions in the given mrv set. If such a term is not present in the mrv set (i.e.,  
22 all terms converge to infinity instead of zero), the relation  $f(x) \sim \frac{1}{f(x)}$  can be used. In the  
23 considered case, only item  $\omega = e^{-x}$  can be accepted.

The next step is to rewrite the mrv set in terms of  $\omega = g(x)$ . Every element  $f(x)$  of the mrv set is rewritten as  $A\omega^c$ , where

$$c = \lim_{x \rightarrow \infty} \frac{\log f(x)}{\log g(x)}, \quad A = e^{\log f - c \log g} \quad (3)$$

Note that this step includes calculation of more simple limits, for instance

$$\lim_{x \rightarrow \infty} \frac{\log e^{x+2e^{-x}}}{\log e^{-x}} = \lim_{x \rightarrow \infty} \frac{x + 2e^{-x}}{-x} = -1 \quad (4)$$

24 In this example we obtain the rewritten mrv set:  $\{\frac{1}{\omega}, \omega, \frac{1}{\omega} e^{2\omega}\}$ . This can be done in SymPy with

```
25 >>> from sympy.series.gruntz import mrv, rewrite
26 >>> m = mrv(exp(x+2*exp(-x))-exp(x) + 1/x, x)
27 >>> w = Symbol('w')
28 >>> rewrite(m[1], m[0], x, w)[0]
29 1/x + exp(2*w)/w - 1/w
```

Then the rewritten subexpressions are substituted back into  $f(x)$  in (2) and the result is expanded with respect to  $\omega$ :

$$f(x) = \frac{1}{x} - \frac{1}{\omega} + \frac{1}{\omega} e^{2\omega} = 2 + \frac{1}{x} + 2\omega + O(\omega^2) \quad (5)$$

Since  $\omega$  is from the mrv set, then in the limit as  $x \rightarrow \infty$ ,  $\omega \rightarrow 0$ , and so  $2\omega + O(\omega^2) \rightarrow 0$  in (5):

$$f(x) = \frac{1}{x} - \frac{1}{\omega} + \frac{1}{\omega} e^{2\omega} = 2 + \frac{1}{x} + 2\omega + O(\omega^2) \rightarrow 2 + \frac{1}{x} \quad (6)$$

30 In this example the result  $(2 + \frac{1}{x})$  still depends on  $x$ , so the above procedure is repeated until  
31 just a value independent of  $x$  is obtained. This is the final limit. In the above case the limit is 2,  
32 as can be verified by SymPy:

```
33 >>> limit(exp(x+2*exp(-x))-exp(x) + 1/x, x, oo)
34 2
```

In general, when  $f(x)$  is expanded in terms of  $\omega$ , the following is obtained:

$$f(x) = \underbrace{O\left(\frac{1}{\omega^3}\right)}_{\infty} + \underbrace{\frac{C_{-2}(x)}{\omega^2}}_{\infty} + \underbrace{\frac{C_{-1}(x)}{\omega}}_{\infty} + C_0(x) + \underbrace{C_1(x)\omega}_0 + \underbrace{O(\omega^2)}_0 \quad (7)$$

35 The positive powers of  $\omega$  are zero. If there are any negative powers of  $\omega$ , then the result of the  
36 limit is infinity, otherwise the limit is equal to  $\lim_{x \rightarrow \infty} C_0(x)$ . The expression  $C_0(x)$  is always simpler  
37 than original  $f(x)$ , same is true for limits, arising in the rewrite stage (3), so the algorithm  
38 converges. A proof of this and further details on the algorithm are given in Gruntz's PhD  
39 thesis [7].

## 40 2 SERIES

### 41 2.1 Series Expansion

42 SymPy is able to calculate the symbolic series expansion of an arbitrary series or expression  
43 involving elementary and special functions and multiple variables. For this it has two different  
44 implementations: the `series` method and Ring Series.

45 The first approach stores a series as an instance of the `Expr` class. Each function has its  
46 specific implementation of its expansion, which is able to evaluate the Puiseux series expansion  
47 about a specified point. For example, consider a Taylor expansion about 0:

```

48 >>> series(sin(x+y) + cos(x*y), x, 0, 2)
49 1 + sin(y) + x*cos(y) + 0(x**2)

```

50 The newer and much faster approach called Ring Series makes use of the fact that a truncated  
51 Taylor series is simply a polynomial. Correspondingly, they may be represented by sparse  
52 polynomials which perform well in a under a wide range of cases. Ring Series also gives the user  
53 the freedom to choose the type of coefficients to use, resulting in faster operations on certain  
54 types.

55 For this, several low-level methods for expansion of trigonometric, hyperbolic and other  
56 elementary operations (like series inversion, calculating the  $n$ th root, etc.) are implemented  
57 using variants of the Newton Method [Brent and Zimmermann]. All these support Puiseux series  
58 expansion. The following example demonstrates the use of an elementary function that calculates  
59 the Taylor expansion of the sine of a series.

```

60 >>> from sympy.polys.ring_series import rs_sin
61 >>> R, t = ring('t', QQ)
62 >>> rs_sin(t**2 + t, t, 5)
63 -1/2*t**4 - 1/6*t**3 + t**2 + t

```

64 The function `sympy.polys.rs_series` makes use of these elementary functions to expand  
65 an arbitrary SymPy expression. It does so by following a recursive strategy of expanding the  
66 lowermost functions first and then composing them recursively to calculate the desired expansion.  
67 Currently, it only supports expansion about 0 and is under active development. Ring Series is  
68 several times faster than the default implementation with the speed difference increasing with  
69 the size of the series. The `sympy.polys.rs_series` takes as input any SymPy expression and  
70 hence there is no need to explicitly create a polynomial ring. An example demonstrating its use:

```

71 >>> from sympy.polys.ring_series import rs_series
72 >>> from sympy.abc import a, b
73 >>> rs_series(sin(a + b), a, 4)
74 -1/2*(sin(b))*a**2 + (sin(b)) - 1/6*a**3*(cos(b)) + a*(cos(b))

```

## 75 2.2 Formal Power Series

76 SymPy can be used for computing the formal power series of a function. The implementation  
77 is based on the algorithm described in the paper on formal power series [8]. The advantage of  
78 this approach is that an explicit formula for the coefficients of the series expansion is generated  
79 rather than just computing a few terms.

80 The following example shows how to use `fps`:

```

81 >>> f = fps(sin(x), x, x0=0)
82 >>> f.truncate(6)
83 x - x**3/6 + x**5/120 + 0(x**6)
84 >>> f[15]
85 -x**15/1307674368000

```

## 86 2.3 Fourier Series

87 SymPy provides functionality to compute Fourier series of a function using the `fourier_series`  
88 function:

```

89 >>> L = symbols('L')
90 >>> expr = 2 * (Heaviside(x/L) - Heaviside(x/L - 1)) - 1
91 >>> f = fourier_series(expr, (x, 0, 2*L))
92 >>> f.truncate(3)
93 4*sin(pi*x/L)/pi + 4*sin(3*pi*x/L)/(3*pi) + 4*sin(5*pi*x/L)/(5*pi)

```

## 94 **3 LOGIC**

95 SymPy supports construction and manipulation of boolean expressions through the `sympy.logic`  
96 module. SymPy symbols can be used as propositional variables and subsequently be replaced  
97 with `True` or `False` values. Many functions for manipulating boolean expressions have been  
98 implemented in the `logic` module.

### 99 **3.1 Constructing boolean expressions**

100 A boolean variable can be declared as a SymPy `Symbol`. Python operators `&`, `|` and `~` are  
101 overridden when using SymPy objects to use the SymPy functionality for logical `And`, `Or`, and  
102 `Not`. Other logic functions are also integrated into SymPy, including `Xor` and `Implies`, which are  
103 constructed with `^` and `>>`, respectively. Expressions can therefore be constructed either by using  
104 the shortcut operator notation or by directly creating the relevant objects: `And()`, `Or()`, `Not()`,  
105 `Xor()`, `Implies()`, `Nand()`, `Nor()`, etc.:

```
106 >>> e = (x & y) | z
107 >>> e.subs({x: True, y: True, z: False})
108 True
```

### 109 **3.2 CNF and DNF**

110 Any boolean expression can be converted to conjunctive normal form, disjunctive normal form,  
111 or negation normal form. The API also exposes methods to check if a boolean expression is in  
112 any of the aforementioned forms.

```
113 >>> from sympy.logic.boolalg import is_dnf, is_cnf
114 >>> to_cnf((x & y) | z)
115 And(Or(x, z), Or(y, z))
116 >>> to_dnf(x & (y | z))
117 Or(And(x, y), And(x, z))
118 >>> is_cnf((x | y) & z)
119 True
120 >>> is_dnf((x & y) | z)
121 True
```

### 122 **3.3 Simplification and Equivalence**

123 The `sympy.logic` module supports simplification of given boolean expression by making deductions  
124 from the expression. Equivalence of two logical expressions can also be checked. In the case of  
125 equivalence, the function `bool_map` can be used to show which variables of the first expression  
126 correspond to which variables of the second one.

```
127 >>> a, b, c = symbols('a b c')
128 >>> e = a & (~a | ~b) & (a | c)
129 >>> simplify(e)
130 And(Not(b), a)
131 >>> e1 = a & (b | c)
132 >>> e2 = (x & y) | (x & z)
133 >>> bool_map(e1, e2)
134 (And(Or(b, c), a), {a: x, b: y, c: z})
```

### 135 **3.4 SAT solving**

136 The module also supports satisfiability (SAT) checking of a given boolean expression. If an  
137 expression is satisfiable, it is possible to return a variable assignment which satisfies it. The  
138 API also supports listing all possible assignments. The SAT solver has a clause learning DPLL  
139 algorithm implemented with a watch literal scheme and VSIDS heuristic [11].

```
140 >>> satisfiable(a & (~a | b) & (~b | c) & ~c)
141 False
142 >>> satisfiable(a & (~a | b) & (~b | c) & c)
143 {a: True, b: True, c: True}
```

## 144 4 DIOPHANTINE EQUATIONS

145 Diophantine equations play a central role in number theory. A Diophantine equation has the  
146 form,  $f(x_1, x_2, \dots, x_n) = 0$  where  $n \geq 2$  and  $x_1, x_2, \dots, x_n$  are integer variables. If there are  $n$   
147 integers  $a_1, a_2, \dots, a_n$  such that  $x_1 = a_1, x_2 = a_2, \dots, x_n = a_n$  satisfies the above equation, the  
148 equation is said to be solvable.

149 Currently, the following five types of Diophantine equations can be solved using SymPy's  
150 Diophantine module ( $a_1, \dots, a_{n+1}$ ,  $a$ ,  $b$ ,  $c$ ,  $d$ ,  $e$ ,  $f$ , and  $k$  are explicitly given rational constants):

- 151 • Linear Diophantine equations:  $a_1x_1 + a_2x_2 + \dots + a_nx_n = b$
- 152 • General binary quadratic equation:  $ax^2 + bxy + cy^2 + dx + ey + f = 0$
- 153 • Homogeneous ternary quadratic equation:  $ax^2 + by^2 + cz^2 + dxy + eyz + fzx = 0$
- 154 • Extended Pythagorean equation:  $a_1x_1^2 + a_2x_2^2 + \dots + a_nx_n^2 = a_{n+1}x_{n+1}^2$
- 155 • General sum of squares:  $x_1^2 + x_2^2 + \dots + x_n^2 = k$

156 The diophantine function factors the equation it is given (if possible), solves each factor  
157 separately, and combines the results to give a final solution set. The following examples illustrate  
158 some of the basic functionalities of the Diophantine module.

```
159 >>> from sympy.solvers.diophantine import *
160 >>> diophantine(2*x + 3*y - 5)
161 set([(3*t_0 - 5, -2*t_0 + 5)])
162
163 >>> diophantine(2*x + 4*y - 3)
164 set()
165
166 >>> diophantine(x**2 - 4*x*y + 8*y**2 - 3*x + 7*y - 5)
167 set([(2, 1), (5, 1)])
168
169 >>> diophantine(x**2 - 4*x*y + 4*y**2 - 3*x + 7*y - 5)
170 set([(-2*t**2 - 7*t + 10, -t**2 - 3*t + 5)])
171
172 >>> diophantine(3*x**2 + 4*y**2 - 5*z**2 + 4*x*y - 7*y*z + 7*z*x)
173 set([(-16*p**2 + 28*p*q + 20*q**2,
174 3*p**2 + 38*p*q - 25*q**2,
175 4*p**2 - 24*p*q + 68*q**2)])
176
177 >>> x1, x2, x3, x4, x5, x6 = symbols('x1, x2, x3, x4, x5, x6')
178 >>> diophantine(9*x1**2 + 16*x2**2 + x3**2 + 49*x4**2 + 4*x5**2 - 25*x6**2)
179 set([(70*t1**2 + 70*t2**2 + 70*t3**2 + 70*t4**2 - 70*t5**2, 105*t1*t5,
180 420*t2*t5, 60*t3*t5, 210*t4*t5,
181 42*t1**2 + 42*t2**2 + 42*t3**2 + 42*t4**2 + 42*t5**2)])
182
183 >>> a, b, c, d = symbols('a:d')
184 >>> diophantine(a**2 + b**2 + c**2 + d**2 - 23)
185 set([(2, 3, 3, 1)])
```

## 186 5 SETS

187 SymPy supports representation of a wide variety of mathematical sets. This is achieved by first  
188 defining abstract representations of atomic set classes and then combining and transforming  
189 them using various set operations.

190 Each of the set classes inherits from the base class `Set` and defines methods to check  
191 membership and calculate unions, intersections, and set differences. When these methods are  
192 not able to evaluate to atomic set classes, they are represented as abstract unevaluated objects.

193 SymPy has the following atomic set classes:

- 194 • `EmptySet` represents the empty set  $\emptyset$ .
- 195 • `UniversalSet` is an abstract “universal set” of which everything is a member. The union of  
196 the universal set with any set gives the universal set and the intersection gives the other  
197 set itself.
- 198 • `FiniteSet` is functionally equivalent to Python’s built in `set` object. Its members can be  
199 any SymPy object including other sets.
- 200 • `Integers` represents the set of integers  $\mathbb{Z}$ .
- 201 • `Naturals` represents the set of natural numbers  $\mathbb{N}$ , i.e., the set of positive integers.
- 202 • `Naturals0` represents the set of whole numbers  $\mathbb{N}_0$ , which are all the non-negative integers.
- 203 • `Range` represents a range of integers. A range is defined by specifying a start value, an end  
204 value, and a step size. The enumeration of a `Range` object is functionally equivalent to  
205 Python’s `range` except it supports infinite endpoints, allowing the representation of infinite  
206 ranges.
- 207 • `Interval` represents an interval of real numbers. It is defined by giving the start and the  
208 end points and by specifying if the interval is open or closed on the respective ends.

209 Other than unevaluated classes of `Union`, `Intersection`, and `Complement` operations, SymPy  
210 has the following set classes.

- 211 • `ProductSet` defines the Cartesian product of two or more sets. The product set is useful  
212 when representing higher dimensional spaces. For example, to represent a three-dimensional  
213 space, SymPy uses the Cartesian product of three real sets.
- 214 • `ImageSet` represents the image of a function when applied to a particular set. The image  
215 set of a function  $F$  with respect to a set  $S$  is  $\{F(x) \mid x \in S\}$ . SymPy uses image sets to  
216 represent sets of infinite solutions of equations such as  $\sin(x) = 0$ .
- 217 • `ConditionSet` represents a subset of a set whose members satisfy a particular condition.  
218 The subset of set  $S$  given by the condition  $H$  is  $\{x \mid H(x), x \in S\}$ . SymPy uses condition  
219 sets to represent the set of solutions of equations and inequalities, where the equation or  
220 the inequality is the condition and the set is the domain over which it is being solved.

221 A few other classes are implemented as special cases of the classes described above. The set of  
222 real numbers, `Reals`, is implemented as a special case of `Interval`. `ComplexRegion` is implemented  
223 as a special case of `ImageSet`. `ComplexRegion` supports both polar and rectangular representation  
224 of regions on the complex plane.

## 225 6 STATISTICS

226 The `sympy.stats` module provides random variable types and methods for computing of statistical  
227 properties of expressions involving random variables, which can be either continuous or discrete,  
228 the latter ones being further divided into finite and infinite. The variables are associated with  
229 probability densities on corresponding domains and internally defined in terms of probability  
230 spaces. Apart from the possibility of defining the random variables from user supplied density  
231 distribution, SymPy provides definitions of most common distributions, including `Uniform`,  
232 `Poisson`, `Normal`, `Binomial`, `Bernoulli`, and many others.

233 Properties of random expressions can be calculated using, e.g., `expectation` (abbreviated `E`)  
234 and `variance` to calculate expectation and variance. Internally, these functions generate integrals  
235 and summations, which are automatically evaluated. The evaluation can be suppressed using  
236 `evaluate=False` keyword argument.

237 Conditions on random variables can be defined with inequalities, equalities, and logical  
238 operators and their overall probabilities are obtained using `P`. The features can be illustrated on  
239 a model of two dice throws:

```

240 >>> from sympy.stats import Die, P, E
241 >>> X, Y = Die("X"), Die("Y")
242 >>> P(Eq(X, 6) & Eq(Y, 6))
243 1/36
244 >>> P(X>Y)
245 5/12

```

246 The conditions can also be supplied as a second parameter to E, P, and other methods to calculate  
 247 the property given the condition:

```

248 >>> E(X, X+Y<5)
249 5/3

```

250 Using the facilities of the `sympy.stats` module, one can, for example, calculate the well known  
 251 properties of maxwellian velocity distribution

```

252 >>> from sympy.stats import Maxwell, density
253 >>> kT, m, x = symbols("kT m x", positive=True)
254 >>> v = Maxwell("v", sqrt(kT/m))
255 >>> E(v) # mean velocity
256 2*sqrt(2)*sqrt(kT)/(sqrt(pi)*sqrt(m))
257 >>> E(v, evaluate=False) # unevaluated mean velocity
258 Integral(sqrt(2)*m**(3/2)*v**3*exp(-m*v**2/(2*kT))/(sqrt(pi)*kT**(3/2)),
259 (v, 0, oo))
260 >>> E(m*v**2/2) # mean energy
261 3*kT/2
262 >>> solve(density(v)(x).diff(x), x)[0] # most probable velocity
263 sqrt(2)*sqrt(kT)/sqrt(m)

```

264 More information on the `sympy.stats` module can be found in [13].

## 265 7 CATEGORY THEORY

266 SymPy includes a module for dealing with categories—abstract mathematical objects representing  
 267 classes of structures as classes of objects (points) and morphisms (arrows) between the objects.  
 268 The module was designed with the following two goals in mind:

- 269 1. automatic typesetting of diagrams given by a collection of objects and of morphisms  
 270 between them, and
- 271 2. specification and semi-automatic derivation of properties using commutative diagrams.

272 As of version 1.0, SymPy only implements the first goal, while a partially working draft  
 273 of implementation of the second goal is available at <https://github.com/scolobb/sympy/tree/ct4-commutativity>.  
 274

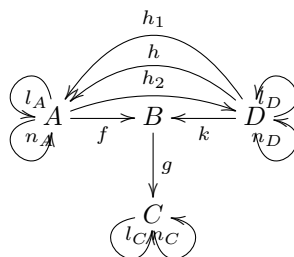
275 In order to achieve the two goals, the module `sympy.categories` defines several classes  
 276 representing some of the essential concepts: objects, morphisms, categories, and diagrams. In  
 277 category theory, the inner structure of objects is often discarded in the favor of studying the  
 278 properties of morphisms, so the class `Object` is essentially a synonym of the class `Symbol`. There  
 279 are several morphism classes which do not have a particular internal structure either, though an  
 280 exception is `CompositeMorphism`, which essentially stores a list of morphisms.

281 The class `Diagram` captures the properties of morphisms. This class stores a family of  
 282 morphisms, the corresponding source and target objects, and, possibly, some properties of the  
 283 morphisms. Generally, no restrictions are imposed on what the properties may be—for example,  
 284 one might use strings of the form “forall”, “exists”, “unique”, etc. Furthermore, the morphisms  
 285 of a diagram are grouped into *premises* and *conclusions* in order to be able to represent logical  
 286 implications of the form “for a collection of morphisms  $P$  with properties  $p : P \rightarrow \Omega$  (the premises),  
 287 there exists a collection of morphisms  $C$  with properties  $c : C \rightarrow \Omega$  (the conclusions)”, where  $\Omega$  is

288 the universal collection of properties. Finally, the class `Category` includes a collection of diagrams  
 289 which are deemed commutative and which therefore define the properties of this category.

290 Automatic typesetting of diagrams takes a `Diagram` and produces L<sup>A</sup>T<sub>E</sub>X code using the `Xy-pic`  
 291 package. Typesetting is done in two stages: layout and generation of `Xy-pic` code. The layout  
 292 stage is taken care of by the class `DiagramGrid`, which takes a `Diagram` and lays out the objects  
 293 in a grid, trying to reduce the average length of the arrows in the final picture. By default,  
 294 `DiagramGrid` uses a series of triangle-based heuristics to produce a rectangular grid. A linear  
 295 layout can also be imposed. Furthermore, groups of objects can be given; in this case, the groups  
 296 will be treated as atomic cells, and the member objects will be typeset independently of the  
 297 other objects.

298 The second phase of diagram typesetting consists in actually drawing the picture and is  
 299 carried out by the class `XypicDiagramDrawer`. An example of a diagram automatically typeset by  
`DiagramGrid` and `XypicDiagramDrawer` is given in Figure 1.



**Figure 1.** An automatically typeset commutative diagram

300

301 As far as the second main goal of the module is concerned, the principal idea consists in  
 302 automatically deciding whether a diagram is commutative or not, given a collection of “axioms”:  
 303 diagrams known to be commutative. The implementation is based on graph embeddings (injective  
 304 maps): whenever an embedding of a commutative diagram into a given diagram is found, one  
 305 concludes that the subdiagram is commutative. Deciding commutativity of the whole diagram is  
 306 therefore based (theoretically) on finding a “cover” of the target diagram by embeddings of the  
 307 axioms. The naïve implementation proved to be prohibitively slow; a better optimized version is  
 308 therefore in order, as well as application of heuristics.

## 309 8 SYMPY GAMMA

310 SymPy Gamma is a simple web application that runs on Google App Engine. It executes and  
 311 displays the results of SymPy expressions as well as additional related computations, in a fashion  
 312 similar to that of Wolfram|Alpha. For instance, entering an integer will display its prime factors,  
 313 digits in the base-10 expansion, and a factorization diagram. Entering a function will display its  
 314 docstring; in general, entering an arbitrary expression will display its derivative, integral, series  
 315 expansion, plot, and roots.

316 SymPy Gamma also has several features beyond just computing the results using SymPy.

- 317 • SymPy Gamma displays integration and differentiation steps in detail, which can be viewed  
 318 in Figure 2:



Integral Steps:  
`integrate(tan(x), x)`

Fullscreen

1. Rewrite the integrand:

$$\tan(x) = \frac{\sin(x)}{\cos(x)}$$

2. Let  $u = \cos(x)$ .

Then let  $du = -\sin(x)dx$  and substitute  $du$ :

$$\int -\frac{1}{u} du$$

A. The integral of a constant times a function is the constant times the integral of the function:

$$\int -\frac{1}{u} du = - \int \frac{1}{u} du$$

I. The integral of  $\frac{1}{u}$  is  $\log(u)$ .

So, the result is:  $-\log(u)$

Now substitute  $u$  back in:

$$-\log(\cos(x))$$

3. Add the constant of integration:

$$-\log(\cos(x)) + \text{constant}$$

The answer is:

$$-\log(\cos(x)) + \text{constant}$$

319

320

**Figure 2.** Integral steps of  $\tan(x)$

321

- SymPy Gamma displays the factor tree diagrams for different numbers.

322

- SymPy Gamma saves user search queries, and offers many such similar features for free, which Wolfram|Alpha only offers to its paid users.

323

324 Every input query from the user on SymPy Gamma is first parsed by its own parser capable of  
325 handling several different forms of function names which SymPy as a library does not support.  
326 For instance, SymPy Gamma supports queries like `sin x`, whereas SymPy will only recognise  
327 `sin(x)`.

328

This parser converts the input query to the equivalent SymPy readable code, which is then  
329 processed by SymPy, and the result is finally printed with the built-in  $\LaTeX$  output and rendered  
330 by the SymPy Gamma web application.

## 331 9 SYMPY LIVE

332

SymPy Live is an online Python shell, which uses the Google App Engine to executes SymPy  
333 code. It is integrated in the SymPy documentation examples at <http://docs.sympy.org>.

334

This is accomplished by providing a HTML/JavaScript GUI for entering source code and  
335 visualization of output, and a server that evaluates the requested source code. It is an interactive  
336 AJAX shell that runs SymPy code using Python on the server.

## 10 COMPARISON WITH MATHEMATICA

Wolfram Mathematica is a popular proprietary CAS that features highly advanced algorithms, has a core written in C++ [16], and interprets its own programming language, Wolfram Language.

Analogous to Lisp S-expressions, Mathematica uses its own style of M-expressions, which are arrays of either atoms or other M-expressions. The first element of the expression identifies the type of the expression and is indexed by zero, and the first argument is indexed starting with one. In SymPy, expression arguments are stored in a Python tuple (that is, an immutable array), while the expression type is identified by the type of the object storing the expression.

Mathematica can associate attributes to its atoms. Attributes may define mathematical properties and behavior of the nodes associated to the atom. In SymPy, the usage of static class fields is roughly similar to Mathematica's attributes, though other programming patterns may also be used to achieve an equivalent behavior such as class inheritance.

Unlike SymPy, Mathematica's expressions are mutable: one can change parts of the expression tree without the need of creating a new object. The mutability of Mathematica expressions allows for a lazy updating of any references to a given data structure.

Products in Mathematica are determined by some built in node types, such as `Times`, `Dot`, and others. `Times` is a representation of the `*` operator, and is always meant to represent a commutative product operator. The other notable product is `Dot`, which represents the `.` operator. This product represents matrix multiplication. It is not commutative. Unlike Mathematica, SymPy determines commutativity with respect to multiplication from the expression type of the factors. Mathematica puts the `Orderless` attribute on the expression type.

Regarding associative expressions, SymPy handles associativity of sums and products by automatically flattening them, Mathematica specifies the `Flat` attribute on the expression type.

Mathematica relies heavily on pattern matching—even the so-called equivalent of function declaration is in reality the definition of a pattern generating an expression tree transformation on input expressions. Mathematica's pattern matching is sensitive to associative, commutative, and one-identity properties of its expression tree nodes. SymPy has various ways to perform pattern matching. All of them play a lesser role in the CAS than in Mathematica and are basically available as a tool to rewrite expressions. The differential equation solver in SymPy somewhat relies on pattern matching to identify differential equation types, but it is envisaged to replace that strategy with analysis of Lie symmetries in the future. Mathematica's real advantage is the ability to add (at runtime) new overloading to the expression builder or specific subnodes. Consider for example:

```
In[1]:= Unprotect[Plus]
Out[1]= {Plus}

In[2]:= Sin[x_]^2 + Cos[y_]^2 := 1

In[3]:= x + Sin[t]^2 + y + Cos[t]^2
Out[3]= 1 + x + y
```

This expression in Mathematica defines a substitution rule that overloads the functionality of the `Plus` node (the node for additions in Mathematica). A symbol with a trailing underscore is treated as a wildcard. Although one may wish to keep this identity unevaluated, this example clearly illustrates the potential to define one's own immediate transformation rules. In SymPy, the operations constructing the addition node in the expression tree are Python class constructors and cannot be modified at runtime.<sup>1</sup> The way SymPy deals with extending the missing runtime overloadability functionality is by subclassing the node types: subclasses may redefine the class constructor to yield the proper extended functionality.

Unlike SymPy, Mathematica does not support type inheritance or polymorphism [4]. SymPy relies heavily on class inheritance, but for the most part, class inheritance is used to make sure that SymPy objects inherit the proper methods and implement the basic hashing system.

---

<sup>1</sup>Nonetheless, Python supports monkey patching but it is a discouraged programming pattern.

388 While Mathematica interprets nested lists as matrices whenever the sublists have the same  
389 length, matrices in SymPy are a type in their own right, allowing ordinary operators and functions  
390 (like multiplication and exponentiation) to be used as they traditionally are in mathematics.

```
391 >>> exp(Matrix([[1, 1],[0, 2]])) * Matrix([a, b])  
392 Matrix([  
393 [E*a + b*(-E + exp(2))],  
394 [ b*exp(2)]])
```

395 Using the standard multiplication in Mathematica performs an element-wise product and  
396 calling the exponential function `Exp` on a matrix returns an element-wise exponentiation of its  
397 elements.

398 Unevaluated expressions in Mathematica can be achieved in various ways, most commonly  
399 with the `HoldForm` or `Hold` nodes, that block the evaluation of subnodes by the parser. Such a  
400 node cannot be expressed in Python because of greedy evaluation. Whenever needed in SymPy,  
401 it is necessary to add the parameter `evaluate=False` to all subnodes.

402 In Mathematica, the operator `==` returns a boolean whenever it is able to immediately evaluate  
403 the truth of the equality, otherwise it returns an `Equal` expression. In SymPy, `==` means structural  
404 equality and is always guaranteed to return a boolean expression. To express a mathematical  
405 equality in SymPy it is necessary to explicitly construct an instance of the `Equality` class.

406 SymPy, in accordance with Python (and unlike the usual programming convention), uses `**`  
407 to express the power operator, while Mathematica uses the more common `^`.

408 SymPy's use of floating-point numbers is similar to that of most other CASs, including  
409 Maple and Maxima. By contrast, Mathematica uses a form of significance arithmetic [14]  
410 for approximate numbers. This offers further protection against numerical errors, although it  
411 comes with its own set of problems (for a critique of significance arithmetic, see Fateman [4]).  
412 Internally, SymPy's `evalf` method works similarly to Mathematica's significance arithmetic, but  
413 the semantics are isolated from the rest of the system.

## 414 11 OTHER PROJECTS THAT USE SYMPY

415 There are several projects that use SymPy as a library for implementing a part of their function-  
416 ality. Some of them are listed below:

- 417 • **Cadabra**: Cadabra is a CAS designed specifically for the resolution of problems encountered  
418 in field theory.
- 419 • **Octave Symbolic**: The Octave-Forge Symbolic package adds symbolic calculation features  
420 to GNU Octave. These include common CAS tools such as algebraic operations, calculus,  
421 equation solving, Fourier and Laplace transforms, variable precision arithmetic, and other  
422 features.
- 423 • **SymPy.jl**: Provides a Julia interface to SymPy using PyCall.
- 424 • **Mathics**: Mathics is a free, general-purpose online CAS featuring Mathematica compatible  
425 syntax and functions. It is backed by highly extensible Python code, relying on SymPy for  
426 most mathematical tasks.
- 427 • **Mathpix**: An iOS App, that detects handwritten math as input, and uses SymPy Gamma  
428 to evaluate the math input and generate the relevant steps to solve the problem.
- 429 • **IKFast**: IKFast is a robot kinematics compiler provided by OpenRAVE. It analytically  
430 solves robot inverse kinematics equations and generates optimized C++ files. It uses  
431 SymPy for its internal symbolic mathematics.
- 432 • **Sage**: A CAS, visioned to be a viable free open source alternative to Magma, Maple,  
433 Mathematica and MATLAB. Sage includes many open source mathematical libraries,  
434 including SymPy.

- 435 • **SageMathCloud**: SageMathCloud is a web-based cloud computing and course manage-  
436 ment platform for computational mathematics.
- 437 • **PyDy**: Multibody Dynamics with Python.
- 438 • **galgebra**: Geometric algebra (previously `sympy.galgebra`).
- 439 • **yt**: Python package for analyzing and visualizing volumetric data (`yt.units` uses SymPy).
- 440 • **SfePy**: Simple finite elements in Python, see section 11.1.
- 441 • **Quameon**: Quantum Monte Carlo in Python.
- 442 • **Lcapy**: Experimental Python package for teaching linear circuit analysis.
- 443 • **Quantum Programming in Python**: Quantum 1D Simple Harmonic Oscillator and  
444 Quantum Mapping Gate.
- 445 • **LaTeX Expression project**: Easy  $\text{\LaTeX}$  typesetting of algebraic expressions in symbolic  
446 form with automatic substitution and result computation.
- 447 • **Symbolic statistical modeling**: Adding statistical operations to complex physical  
448 models.

### 449 11.1 SfePy

450 **SfePy** (Simple finite elements in Python), cf. [3], is a Python package for solving partial  
451 differential equations (PDEs) in 1D, 2D and 3D by the finite element (FE) method [17]. SymPy  
452 is used within this package mostly for code generation and testing, namely:

- 453 • generation of the hierarchical FE basis module, involving generation and symbolic differenti-  
454 ation of 1D Legendre and Lobatto polynomials, constructing the FE basis polynomials [15]  
455 and generating the C code;
- 456 • generation of symbolic conversion formulas for various groups of elastic constants [6]:  
457 provide any two of the Young’s modulus, Poisson’s ratio, bulk modulus, Lamé’s first  
458 parameter, shear modulus (Lamé’s second parameter) or longitudinal wave modulus and  
459 get the other ones;
- 460 • simple physical unit conversions, generation of consistent unit sets;
- 461 • testing FE solutions using method of manufactured (analytical) solutions: the differential  
462 operator of a PDE is symbolically applied and a symbolic right-hand side is created,  
463 evaluated in quadrature points, and subsequently used to obtain a numerical solution that  
464 is then compared to the analytical one;
- 465 • testing accuracy of 1D, 2D and 3D numerical quadrature formulas (cf. [1]) by generating  
466 polynomials of suitable orders, integrating them, and comparing the results with those  
467 obtained by the numerical quadrature.

## 468 12 TENSORS

469 Ongoing work to provide the capabilities of tensor computer algebra has so far produced the  
470 `tensor` module. It comprises three submodules whose purposes are quite different: `sympy.`  
471 `tensor.indexed` and `sympy.tensor.indexed_methods` support indexed symbols, `sympy.tensor.`  
472 `array` contains facilities to operate on symbolic  $N$ -dimensional arrays, and finally `sympy.tensor.`  
473 `tensor` is used to define abstract tensors. The abstract tensors submodule is inspired by xAct [10]  
474 and Cadabra [12]. Canonicalization based on the Butler-Portugal [9] algorithm is supported in  
475 SymPy. Tensor support in SymPy is currently limited to polynomial tensor expressions.

## 476 13 NUMERICAL SIMPLIFICATION

477 The `nsimplify` function in SymPy (a wrapper of `identify` in `mpmath`) attempts to find a simple  
478 symbolic expression that evaluates to the same numerical value as the given input. It works  
479 by applying a few simple transformations (including square roots, reciprocals, logarithms and  
480 exponentials) to the input and, for each transformed value, using the PSLQ algorithm [5] to  
481 search for a matching algebraic number or optionally a linear combination of user-provided base  
482 constants (such as  $\pi$ ).

```
483 >>> t = 1 / (sin(pi/5)+sin(2*pi/5)+sin(3*pi/5)+sin(4*pi/5))**2
484 >>> nsimplify(t)
485 -2*sqrt(5)/5 + 1
486 >>> nsimplify(pi, tolerance=0.01)
487 22/7
488 >>> nsimplify(1.783919626661888, [pi], tolerance=1e-12)
489 pi/(-1/3 + 2*pi/3)
```

## 490 14 EXAMPLES

### 491 14.1 Simplification

- 492 • `expand`:

```
493 >>> expand((x + y)**3)
494 x**3 + 3*x**2*y + 3*x*y**2 + y**3
```

- 495 • `factor`:

```
496 >>> factor(x**3 + 3*x**2*y + 3*x*y**2 + y**3)
497 (x + y)**3
```

- 498 • `collect`:

```
499 >>> collect(y*x**2 + 3*x**2 - x*y + x - 1, x)
500 x**2*(y + 3) + x*(-y + 1) - 1
```

- 501 • `cancel`:

```
502 >>> cancel((x**2 + 2*x + 1)/(x**2 - 1))
503 (x + 1)/(x - 1)
```

- 504 • `apart`:

```
505 >>> apart((x**3 + 4*x - 1)/(x**2 - 1))
506 x + 3/(x + 1) + 2/(x - 1)
```

- 507 • `trigsimp`:

```
508 >>> trigsimp(cos(x)**2*tan(x) - sin(2*x))
509 -sin(2*x)/2
```

### 510 14.2 Polynomials

- 511 • `Factorization`:

```
512 >>> t = symbols('t')
513 >>> f = (2115*x**4*y + 45*x**3*z**3*t**2 - 45*x**3*t**2 -
514 ...      423*x*y**4 - 47*x*y**3 + 141*x*y*z**3 + 94*x*y*z*t -
515 ...      9*y**3*z**3*t**2 + 9*y**3*t**2 - y**2*z**3*t**2 +
```

```

516     ...     y**2*t**2 + 3*z**6*t**2 + 2*z**4*t**3 - 3*z**3*t**2 -
517     ...     2*z*t**3)
518 >>> factor(f)
519 (t**2*z**3 - t**2 + 47*x*y)*(2*t*z + 45*x**3 - 9*y**3 - y**2 +
520     3*z**3)

```

- Gröbner bases:

```

522 >>> x0, x1, x2 = symbols('x:3')
523 >>> I = [x0 + 2*x1 + 2*x2 - 1,
524     ...     x0**2 + 2*x1**2 + 2*x2**2 - x0,
525     ...     2*x0*x1 + 2*x1*x2 - x1]
526 >>> groebner(I, order='lex')
527 GroebnerBasis([7*x0 - 420*x2**3 + 158*x2**2 + 8*x2 - 7,
528     7*x1 + 210*x2**3 - 79*x2**2 + 3*x2,
529     84*x2**4 - 40*x2**3 + x2**2 + x2], x0, x1, x2, domain='ZZ',
530     order='lex')

```

- Root isolation:

```

532 >>> f = 7*z**4 - 19*z**3 + 20*z**2 + 17*z + 20
533 >>> intervals(f, all=True, eps=0.001)
534 ([],
535     [((-425/1024 - 625*I/1024, -1485/3584 - 2185*I/3584), 1),
536     ((-425/1024 + 2185*I/3584, -1485/3584 + 625*I/1024), 1),
537     ((3175/1792 - 2605*I/1792, 1815/1024 - 10415*I/7168), 1),
538     ((3175/1792 + 10415*I/7168, 1815/1024 + 2605*I/1792), 1)])

```

### 14.3 Solvers

- Single solution:

```

541 >>> solveset(x - 1, x)
542 {1}

```

- Finite solution set, quadratic equation:

```

544 >>> solveset(x**2 - pi**2, x)
545 {-pi, pi}

```

- No solution:

```

547 >>> solveset(1, x)
548 EmptySet()

```

- Interval solution:

```

550 >>> solveset(x**2 - 3 > 0, x, domain=S.Reals)
551 (-oo, -sqrt(3)) U (sqrt(3), oo)

```

- Infinitely many solutions:

```

553 >>> solveset(x - x, x, domain=S.Reals)
554 (-oo, oo)
555 >>> solveset(x - x, x, domain=S.Complexes)
556 S.Complexes

```

- 557 • Linear systems (linsolve)

```

558 >>> A = Matrix([[1, 2, 3], [4, 5, 6], [7, 8, 10]])
559 >>> b = Matrix([3, 6, 9])
560 >>> linsolve((A, b), x, y, z)
561 {(-1, 2, 0)}
562 >>> linsolve(Matrix([[1, 1, 1, 1], [1, 1, 2, 3]]), (x, y, z))
563 {(-y - 1, y, 2)}

```

564 Below are examples of solve applied to problems not yet handled by solveset.

- 565 • Nonlinear (multivariate) system of equations (the intersection of a circle and a parabola):

```

566 >>> solve([x**2 + y**2 - 16, 4*x - y**2 + 6], x, y)
567 [(-2 + sqrt(14), -sqrt(-2 + 4*sqrt(14))),
568  (-2 + sqrt(14), sqrt(-2 + 4*sqrt(14))),
569  (-sqrt(14) - 2, -I*sqrt(2 + 4*sqrt(14))),
570  (-sqrt(14) - 2, I*sqrt(2 + 4*sqrt(14)))]

```

- 571 • Transcendental equations:

```

572 >>> solve((x + log(x))**2 - 5*(x + log(x)) + 6, x)
573 [LambertW(exp(2)), LambertW(exp(3))]
574 >>> solve(x**3 + exp(x))
575 [-3*LambertW((-1)**(2/3)/3)]

```

## 576 14.4 Matrices

- 577 • Matrix expressions

```

578 >>> m, n, p = symbols('m n p', integer=True)
579 >>> R = MatrixSymbol('R', m, n)
580 >>> S = MatrixSymbol('S', n, p)
581 >>> T = MatrixSymbol('T', m, p)
582 >>> U = R*S + 2*T
583 >>> U.shape
584 (m, p)
585 >>> U[0, 1]
586 2*T[0, 1] + Sum(R[0, _k]*S[_k, 1], (_k, 0, n - 1))

```

- 587 • Block Matrices

```

588 >>> n, m, l = symbols('n m l')
589 >>> X = MatrixSymbol('X', n, n)
590 >>> Y = MatrixSymbol('Y', m, m)
591 >>> Z = MatrixSymbol('Z', n, m)
592 >>> B = BlockMatrix([[X, Z], [ZeroMatrix(m, n), Y]])
593 >>> B
594 Matrix([
595  [X, Z],
596  [0, Y]])
597 >>> B[0, 0]
598 X[0, 0]
599 >>> B.shape
600 (m + n, m + n)

```

## 15 REFERENCES

### REFERENCES

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